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Linear wave equations as motions on a Toda lattice

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Abstract. A bijection is defined from the set of motions on the infinite Toda lattice of strings to a set of sequences of linear wave equations in $1+1$ dimensions, the sequences being generated by a generalisation of the classical Darboux map. The bijection is applied to find probably all such wave equations for which characteristic initial data propagate without spreading.

1. Introduction

We shall construct a fruitful correspondence between the set of motions of a particular non-linear dynamical system involving exponential interactions and the set of linear second-order wave equations in $1+1$ dimensions (LSWE). The non-linear system in question is the generalisation of the infinite Toda (1981) lattice wherein Toda's system of second-order non-linear *ordinary* differential equations in time becomes a system of non-linear *wave* equations in $1+1$ dimensions (see Leznov and Saveliev 1981). We will refer to this system as the infinite Toda lattice of strings†. The correspondence will be based on a related pair of bijections $B: M \to W$, $\bar{B}: M \to \bar{W}$, where M is the set of motions of the Toda lattice of strings. To define W and $\bar{W}$ we will first put the LSWE into either one of two normal forms, thus reducing LSWE to either one of two subsets WE, $\overline{WE}$, and then define W and $\bar{W}$ to be certain sets of sequences in WE and $\overline{WE}$, respectively. The sequences will be constructed from a generalisation to wave equations of the classical Darboux (1882) (see Levi *et al* (1984) for a current reference) transformation between Schrödinger equations. The usefulness of $B: M \to W$, $\bar{B}: M \to \bar{W}$ rests on the fact that a great deal is known about the motions of the various types of Toda lattices, on the one hand, and about linear wave equations in $1+1$ dimensions, on the other, and that known results about a subset of either set may map by $B(\bar{B})$, or $B^{-1}(\bar{B}^{-1})$, into new results about a subset of the other.

The focus of this paper will be the application of B and $\bar{B}$ to obtain the complete solution of an interesting question regarding linear wave equations posed by Kundt and Newman (1968, hereafter referred to as KN) by showing it to be the image under B (or $\bar{B}$) of a solved problem concerning the Toda lattice of strings. The unanswered question concerns the characterisation of those LSWE that possess the characteristic propagation property (CPP). This means, briefly, that characteristic initial data generate a field whose support is restricted to field points connected by unbroken characteristics to the support of the initial data. It was shown in KN that a sufficient, and probably

† This seems to us preferable to calling it the two-dimensional Toda lattice, as is commonly done, since the latter has an obvious alternative interpretation.

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necessary, condition for the CPP is the 'double termination' of the 'substitution sequence' generated by the coefficients of the wave equation. It is shown in this paper that the subset of LSWE with doubly terminating substitution sequences is mapped by B^{-1} and $\bar{B}^{-1}$, in two different ways, to precisely the set of motions of finite Toda lattices of strings with both end strings free. This system can be viewed as a special case of the infinite Toda lattice of strings. Since Leznov and Saveliev have given closed form expressions for the general motion of this special system, we are able to write down explicit formulae for the coefficients of all the LSWE with doubly terminating substitution sequences, and thus of probably all LSWE with the CPP. Of no less interest is the fact that the general solutions of this class of LSWE can be given in a simple closed form and that we may in a sense have constructed all LSWE with this attractive property.

We will briefly review relevant results concerning the infinite Toda lattice of strings and the Kundt-Newman substitution sequences in §§ 2 and 3, respectively. In § 4, $B: M \to W$, $\bar{B}: M \to \bar{W}$ will be defined, and they will be applied in § 5 to the construction of the LSWE with doubly terminating substitution sequences. In § 6 we will extract the subset of self-adjoint LSWE with this property and describe the corresponding subset of motions of the Toda lattice of strings. This special case has already been dealt with, without reference to the Toda lattice, by Torrence (1986). The content of § 2 is strictly a recapitulation of known results, while § 3 contains some mildly original elements. The result of § 4 seems to be entirely new and, while formal in nature, joins two ostensibly independent mathematical structures in what promises to be a productive partnership. As evidence for this claim, the application in § 5 appears to give a complete solution to a natural, and hitherto unsolved, problem. In § 6 known results are related to the new ones presented in § 5. There is a concluding § 7.

2. Toda lattices of strings

The infinite Toda lattice is generally pictured as an infinite set of particles distributed along a straight line and experiencing longitudinal oscillations under a nearest-neighbour interaction of exponential form. If we denote the displacement, however viewed, of the nth particle by y_n, it is easy to see that the equation of motion for that particle, of mass m_n, is given by

$$m_n \frac{d^2}{dt^2} y_n = \Phi'(y_{n+1} - y_n) - \Phi'(y_n - y_{n-1}) \qquad -\infty < n < +\infty \tag{2.1}$$

where Φ is the potential for the nearest-neighbour interaction. Assuming that

$$\Phi(x) = e^{-x} \tag{2.2}$$

corresponding to repulsive exponential forces, putting $m_n = 1$ for all n and denoting the relative displacement by $r_n \equiv y_{n+1} - y_n$, we find that the *infinite Toda lattice* is governed by the infinite system of non-linear ordinary differential equations

$$\frac{d^2}{dt^2} \begin{bmatrix} \vdots \\ r_{n-1} \\ r_n \\ r_{n+1} \\ \vdots \end{bmatrix} = K_\infty \begin{bmatrix} \vdots \\ \exp(-r_{n-1}) \\ \exp(-r_n) \\ \exp(-r_{n+1}) \\ \vdots \end{bmatrix} \tag{2.3}$$

where

$$K_\infty = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ \dots & 0 & -1 & 2 & -1 & 0 & \dots & & \\ & \dots & 0 & -1 & 2 & -1 & 0 & \dots & \\ & & \dots & 0 & -1 & 2 & -1 & 0 & \dots \end{bmatrix}. \tag{2.4}$$

Other types of Toda lattice such as the periodic lattice and the finite lattice with ends free, or fixed, can be usefully seen as special cases of (2.3), and we will consider one of these specialisations in a more general context in what follows.

The extensive study of the Toda lattice has spawned a variety of generalisations, one of which is central to this paper. Imagine each particle of the lattice to be extended into a stretched string, orthogonal to the original array, with the strings experiencing a transverse oscillation in the plane of the new array and subject to a nearest-neighbour exponential coupling. A motion of this system comprises waves on each of the stretched strings. The generalisation of (2.1) appropriate to this case is

$$\partial^2_{uv} y_n = -\Phi'(y_{n+1} - y_n) + \Phi'(y_n - y_{n-1}) \qquad -\infty < n < +\infty \tag{2.5}$$

where Φ is given by (2.2) and the y_n now depend on the two (characteristic) coordinates u and v. The system of partial differential equations governing this infinite Toda lattice of strings is

$$\partial^2_{uv} \begin{bmatrix} \vdots \\ r_{n-1} \\ r_n \\ r_{n+1} \\ \vdots \end{bmatrix} = -K_\infty \begin{bmatrix} \vdots \\ \exp(-r_{n-1}) \\ \exp(-r_n) \\ \exp(-r_{n+1}) \\ \vdots \end{bmatrix} \tag{2.6}$$

where K_∞ is as before. Of particular interest for our work is a specialisation of (2.6). We can obtain a *finite* Toda lattice of strings with *end strings free* by assuming $-y_0 = y_{N+1} = +\infty$ in (2.5), thus forcing the interaction from the left (right) of y_1 (y_N) to zero, so that y_1 and y_N become free end strings. The relative displacements $r_1, \dots, r_{N-1}$ are governed by a double truncation of (2.6):

$$\partial^2_{uv} \begin{bmatrix} r_1 \\ \vdots \\ r_{N-1} \end{bmatrix} = -K_{N-1} \begin{bmatrix} \exp(-r_1) \\ \vdots \\ \exp(-r_{N-1}) \end{bmatrix} \tag{2.7}$$

where K_{N-1} is the $(N-1)\times(N-1)$ matrix

$$K_{N-1} = \begin{bmatrix} 2 & -1 & 0 & & & & & \\ -1 & 2 & -1 & 0 & & & & \\ 0 & -1 & 2 & -1 & 0 & & & \\ & & & \ddots & & & & \\ & & & 0 & -1 & 2 & -1 & 0 \\ & & & & 0 & -1 & 2 & -1 \\ & & & & & 0 & -1 & 2 \end{bmatrix}. \tag{2.8}$$

Given a solution of (2.7), the extraction of the displacements $y_1, \ldots, y_N$ can be accomplished by solving the first of (2.5):

$$\partial_{uv}^2 y_1 = \exp(-r_1) \tag{2.9}$$

for y_1, and then obtaining y_2, y_3, etc, from y_1 and r_1, r_2, etc.

The application, in § 4, of the bijections $B: M \to W$ and $\bar{B}: M \to \bar{W}$ involves the specialisation just discussed, the finite Toda lattice with free end strings. We shall need the remarkable formulae obtained by Leznov and Saveliev that give the general solution of this integrable non-linear dynamical system for any number of strings. We shall not concern ourselves with the derivation of their results, but just state what we shall need later. The non-trivial part of the problem is to obtain a solution of (2.7) containing $2N-2$ arbitrary single-valued functions; solving (2.9) is then clearly trivial and brings the total number of arbitrary one-variable functions to the required $2N$, for the system of N second-order equations for $y_1, \ldots, y_N$. We first define

$$\begin{bmatrix} \tau_1 \\ \vdots \\ \tau_{N-1} \end{bmatrix} = -K_{N-1}^{-1} \begin{bmatrix} r_1 \\ \vdots \\ r_{N-1} \end{bmatrix} \tag{2.10}$$

and it is easy to see that (2.7) is equivalent to

$$\partial_{uv}^2 \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_{N-1} \end{bmatrix} = \begin{bmatrix} \exp(K_{N-1}\boldsymbol{\tau})_1 \\ \vdots \\ \exp(K_{N-1}\boldsymbol{\tau})_{N-1} \end{bmatrix} \tag{2.11}$$

where $\boldsymbol{\tau}$ stands for the vector with components $\tau_1, \ldots, \tau_{N-1}$. It is (2.11) with which Leznov and Saveliev chose to work. In the simplest case, $N=2$, (2.11) reduces to

$$\partial_{uv}^2 \tau_1 = \exp(2\tau_1) \tag{2.12}$$

the familiar Liouville equation, whose general solution is easily confirmed to be given by

$$\exp(\tau_1) = \frac{[\varphi_1(u)\varphi_1(v)]^{1/2}}{(\int^u \varphi_1\, \mathrm{d}u_1)(\int^v \varphi_1\, \mathrm{d}v_1) - 1}. \tag{2.13}$$

In order to write down the generalisation of (2.13) for arbitrary N we define

$$\Delta_n[X(u,v)] \equiv \begin{bmatrix} X & X_u & & X^{n-1}_{u\ldots u} \\ X_v & X_{vu} & \ldots & X^{n-1}_{u\ldots uv} \\ \vdots & & & \\ X^{n-1}_{v\ldots v} & X^{n-1}_{v\ldots vu} & \ldots & X^{n-1\,n-1}_{v\ldots v\,u\ldots u} \end{bmatrix}. \tag{2.14}$$

Leznov and Saveliev have shown that

$$\exp(-\tau_n) = (-1)^{n(n-1)/2}\Delta_n[X_{N-1}(u,v)] \qquad n = 1, 2, \ldots, N-1 \tag{2.15}$$

yields the general solution of (2.11), where

$$\begin{aligned} \exp(-\tau_1) &= X_{N-1} \\ &= \left[\left(\int^u \varphi_1\, \mathrm{d}u_1 \ldots \int^{u_{N-2}} \varphi_{N-1}\, \mathrm{d}u_{N-1}\right)\left(\int^v \varphi_1\, \mathrm{d}v_1 \ldots \int^{v_{N-2}} \varphi_{N-1}\, \mathrm{d}v_{N-1}\right) - \ldots \right. \\ &\quad \left. + (-1)^{N-2}\left(\int^u \varphi_1\, \mathrm{d}u_1\right)\left(\int^v \varphi_1\, \mathrm{d}v_1\right) + (-1)^{N-1}\right] \end{aligned}$$

$$\times[\varphi_1^{N-1}(u)\varphi_2^{N-2}(u)\dots\varphi_{N-1}(u)\varphi_1^{N-1}(v)\varphi_2^{N-2}(v)\dots\varphi_{N-1}(v)]^{-1/N}. \tag{2.16}$$

Inverting (2.10) we obtain for the general solution of (2.7)

$$\begin{aligned} \exp(r_1) &= -(\Delta_1 X_{N-1})^2/\Delta_2 X_{N-1} \\ &\vdots \\ \exp(r_n) &= -(\Delta_n X_{N-1})^2/(\Delta_{n-1}X_{N-1})(\Delta_{n+1}X_{N-1}) \\ &\vdots \\ \exp(r_{N-1}) &= -(\Delta_{N-1}X_{N-1})^2/\Delta_{N-2}X_{N-1}. \end{aligned} \tag{2.17}$$

Combined with (2.9), (2.17) gives us the general $y_1, \dots, y_N$ for the Toda lattice of N strings, with both end strings free. We can actually obtain explicit formulae for the $y_1, \dots, y_N$ without much difficulty. The definitions of r_n and (2.17) are consistent with the formulae

$$\begin{aligned} \exp(-y_1) &= -\Delta_1 X_{N-1}/\Delta_0 X_{N-1} \\ &\vdots \\ \exp(-y_n) &= (-1)^n \Delta_n X_{N-1}/\Delta_{n-1}X_{N-1} \\ &\vdots \\ \exp(-y_N) &= (-1)^N \Delta_N X_{N-1}/\Delta_{N-1}X_{N-1} \end{aligned} \tag{2.18}$$

where $\Delta_N X_{N-1} = \Delta_0 X_{N-1} \equiv 1$ and $\Delta_1 X_{N-1} = X_{N-1}$ from (2.14). A direct check confirms that (2.18) satisfies (2.9), so we have a solution of (2.7) and (2.9). It is not the general solution, as X_{N-1} defined by (2.16) has only $2N-2$ arbitrary functions, but this is easily remedied. The system (2.16) is easily shown to be invariant in form if we simultaneously perform the transformations

$$\begin{aligned} &\mathrm{d}u'/\mathrm{d}u = \bar{U}(u) && \mathrm{d}v'/\mathrm{d}v = \bar{V}(v) \\ &\exp(y_n') = (\bar{U}\bar{V})^n \exp(y_n) && n = 1, \dots, N. \end{aligned} \tag{2.19}$$

In this way two additional functions can be introduced to obtain the general solution

$$\begin{aligned} \exp(y_1) &= -[\varphi_0(u)\psi_0(v)]\Delta_0 X_{N-1}/\Delta_1 X_{N-1} \\ &\vdots \\ \exp(y_n) &= (-1)^n[\varphi_0(u)\psi_0(v)]^n \Delta_{n-1}X_{N-1}/\Delta_n X_{N-1} \\ &\vdots \\ \exp(y_N) &= (-1)^N[\varphi_0(u)\psi_0(v)]^N \Delta_{N-1}X_{N-1}/\Delta_N X_{N-1} \end{aligned} \tag{2.20}$$

of (2.7) and (2.9).

Some comments regarding the solving of (2.5), (2.7) and (2.9) are in order. From the point of view of a dynamical system it is natural to ask for a solution of the governing equations satisfying prescribed initial conditions. Even with (2.7) and (2.9), where we have the general solution (2.20), it is by no means easy to find the connection between the arbitrary functions in that general solution and given initial conditions. With (2.5), the situation regarding initial conditions is even less clear. Fortunately the dynamically natural problem is secondary to our interest and need not concern us. Suppose instead that we wish to find a solution of (2.5) consistent with the prescribed behaviour of some triple $y_{n-1}(u, v)$, $y_n(u, v)$, $y_{n+1}(u, v)$. Clearly (2.5) will generate, from the given information, the full sequence $\{y_n | n \in Z\}$ and we are done. This will be the proper attitude toward the system (2.5) in the context of § 4. With the finite system governed by (2.7) and (2.9) the situation is a bit different. The end conditions limit our freedom to specify y_n. On the other hand, (2.20) has provided us with the general solution. This important special case, which we will see corresponds to doubly terminating substitution sequences, is the topic in § 6.

3. Substitution sequences and characteristic propagation

It was shown in KN that by using a general coordinate transformation and a factor transformation on the dependent variable we may put every element of LSWE into either right normal form

$$\{\partial_v k(u,v)\partial_u - l(u,v)\}\Psi(u,v) = 0 \tag{3.1R}$$

or left normal form

$$\{\partial_u \bar{k}(u,v)\partial_v - \bar{l}(u,v)\}\bar{\Psi}(u,v) = 0. \tag{3.1L}$$

We will let WE and $\overline{WE}$ represent the sets of right and left normal form wave equations, respectively. Following KN, we introduce a generalisation to the wave equations in WE and $\overline{WE}$ of the classical Darboux map. Given an element of WE suppose we put $j_0 \equiv k$, $j_1 \equiv l$, $\psi_0 \equiv \psi$ and define j_2, ψ_1 by

$$j_2 = j_1[(j_1/j_0) - \partial_{vu}\ln|j_1|] \qquad \partial_v(j_1\psi_1) = j_1\psi_0.$$

It is not hard to confirm (see KN for details) that $(\partial_v j_0 \partial_u - j_1)\psi_0 = 0$ is equivalent to $(\partial_v j_1 \partial_u - j_2)\psi_1 = 0$. More generally, if we inductively define $\{j_n | n \in Z\}$ and $\{\psi_n | n \in Z\}$ by

$$j_{n+1}/j_n = (j_n/j_{n-1}) - \partial^2_{vu}\ln|j_n| \qquad \partial_v(j_n\psi_n) = j_n\psi_{n-1} \tag{3.2R}$$

it follows that

$$(\partial_v j_0 \partial_u - j_1)\psi_0 = 0 \Leftrightarrow (\partial_v j_n \partial_u - j_{n+1})\psi_n = 0 \tag{3.3R}$$

for all integers n, including those less than 0. Following a similar, but not identical, pattern for $\overline{WE}$, we define sequences $\{\bar{j}_n | n \in Z\}$, $\{\bar{\psi}_n | n \in Z\}$ by putting $\bar{j}_0 \equiv \bar{k}$, $\bar{j}_{-1} \equiv \bar{l}$, $\bar{\psi}_0 \equiv \psi$ and replacing (3.2R) by

$$\bar{j}_{n-1}/\bar{j}_n = (\bar{j}_n/\bar{j}_{n+1}) - \partial^2_{uv}\ln|\bar{j}_n| \qquad \partial_u(\bar{j}_n\bar{\psi}_n) = \bar{j}_n\bar{\psi}_{n+1} \tag{3.2L}$$

and it follows that

$$(\partial_u \bar{j}_0 \partial_v - \bar{j}_{-1})\bar{\psi}_0 = 0 \Leftrightarrow (\partial_u \bar{j}_n \partial_v - \bar{j}_{n-1})\bar{\psi}_n = 0 \tag{3.3L}$$

for all integers n.

We have now defined related sequences of j_n, Ψ_n and (equivalent) wave equations w_n in WE, and of $\bar{j}_n$, $\bar{\Psi}_n$ and (equivalent) wave equations $\bar{w}_n$ in $\overline{WE}$. We can (and will) arrange the indexing so that

$$(\partial_v j_n \partial_u - j_{n+1})\Psi_n \sim w_n \tag{3.4R}$$

$$(\partial_u \bar{j}_n \partial_v - j_{n-1})\bar{\Psi}_n \sim \bar{w}_n. \tag{3.4L}$$

In the case where $j_N = 0$ for some $N > 0$ ($1/j_M = 0$ for some $M < 0$) we see that the generation of j_n must terminate to the right (left), while if $\bar{j}_N = 0$, for some $N < 0$ ($1/\bar{j}_M = 0$ for some $M > 0$), the generation of $\bar{j}_n$ must terminate to the left (right), and similarly for the sequences of wave equations. Double termination, resulting in finite sequences, is a possibility that will particularly interest us in § 4. We will refer to the sequences $\{j_n | n \in Z\}$ and $\{w_n | n \in Z\}$, terminating or not, as right substitution sequences, and to those of the form $\{\bar{j}_n | n \in Z\}$, $\{\bar{w}_n | n \in Z\}$ as left substitution sequences. It will be convenient to denote the collections of all such sequences by J and W, and $\bar{J}$ and $\bar{W}$, respectively, and it is clear that W and $\bar{W}$ are in fact partitions of WE and $\overline{WE}$. It is natural to think of a sequence in W, or $\bar{W}$, as being generated by w_0, or $\bar{w}_0$, but it is easy to see that a sequence is equally well generated by any one of its members. Thus no special significance should be attributed to the index 0. In any case, where a particular element in a sequence *is* of special significance we *may* indicate this by denoting it by w_0, or $\bar{w}_0$ (or j_0, or $\bar{j}_0$) but there is no natural candidate for this honour in the general case.

Since an equation in right normal form can always be transformed into an equation in left normal form, and vice versa, we can introduce a natural bijection between W and $\bar{W}$, and between J and $\bar{J}$. It was shown in KN that an element $w_n \sim (\partial_v j_n \partial_u - j_{n+1})\psi_n = 0$ in WE takes the left normal form

$$\bar{w}_n \sim (\partial_u \bar{j}_n \partial_v - \bar{j}_{n-1})\bar{\psi}_n = 0$$

where

$$\bar{j}_n = 1/j_n \qquad \bar{j}_{n-1} = 1/j_{n-1} \qquad \bar{\psi}_n = j_n\psi_n. \tag{3.5R}$$

Similarly $\bar{w}_n \sim (\partial_u \bar{j}_n \partial_v - \bar{j}_{n-1})\bar{\psi}_n = 0$ can be put into the form $w_n \sim (\partial_v j_n \partial_u - j_{n+1})\psi_n = 0$, where

$$j_n = 1/\bar{j}_n \qquad j_{n+1} = 1/\bar{j}_{n+1} \qquad \psi_n = \bar{j}_n\bar{\psi}_n. \tag{3.5L}$$

If we define the mutually inverse bijections

$$R_w : W \to \bar{W} : \{w_n | n \in Z\} \to \{\bar{w}_n | n \in Z\} \tag{3.6R}$$

$$\bar{R}_w : \bar{W} \to W : \{\bar{w}_n | n \in Z\} \to \{w_n | n \in Z\} \tag{3.6L}$$

with (3.5) holding, then the nth element of $R_w(w)$ is the left normal form of the (right normal form equation) w_n, and correspondingly for $\bar{R}_w$. The identification of j_n with w_n, and $\bar{j}_n$ with $\bar{w}_n$, induces the bijections

$$R_j : J \to \bar{J} : \{j_n | n \in Z\} \to \{1/j_n | n \in Z\} \tag{3.7R}$$

$$\bar{R}_j : \bar{J} \to J : \{\bar{j}_n | n \in Z\} \to \{1/\bar{j}_n | n \in Z\}. \tag{3.7L}$$

We will call the image of j under R_j the *reciprocal sequence* of j and for $\bar{j}$ in $\bar{J}$, $\bar{R}_j(\bar{j})$ is the *reciprocal sequence* of $\bar{j}$.

It should not be forgotten that given an element of LSWE its representatives in WE and $\overline{WE}$ are not uniquely determined, as has been carefully discussed in KN. This need not concern us here as far as putting equations into normal form is concerned, as our formal development in this section has referred exclusively to equations already in normal form. The shifting of equations from one normal form to the other, which concerned us in the last paragraph, is also immune to ambiguity due to the condition $j_n\bar{j}_n = 1$ which was implicitly imposed in deriving (3.5). It should be explicitly pointed out that the integrity of the substitution sequences, which are central to this work, would be destroyed if 'gauge' transformations were performed on some of their elements but not others. A definition of such transformations for substitution sequences as a whole could be formulated, but is not essential here and will be omitted.

Everything that we will need in order to establish, in § 4, the bijections $B : M \to W$, $\bar{B} : M \to \bar{W}$ is now in place, but for the sake of the application to be carried out there we must explain the relationship discovered by Kundt and Newman (1968) between the substitution sequences and the characteristic propagation property. For concreteness suppose we consider a wave equation in right normal form w_0 and its left normal form 'equivalent' $\bar{w}_0$, and let us assume that $w=\{w_n|n\in Z\}$ in W double terminates, i.e. that for some $M<0$, $1/j_{M-1}=0$, and for some $N>0$, $j_{N+1}=0$. It is clear that this is equivalent to the double termination of $R_w(w)=\{\bar{w}_n|n\in Z\}$ since $\bar{j}_{M-1}=1/j_{M-1}$ and $1/\bar{j}_{N+1}=j_{N+1}$. Now from $j_{N+1}=0$ we have

$$(\partial_v j_N \partial_u)\psi_N = 0 \Rightarrow \psi_N = a(v) \tag{3.8R}$$

while from $\bar{j}_{M-1}=0$ we have

$$(\partial_u \bar{j}_M \partial_v)\bar{\psi}_M = 0 \Rightarrow \bar{\psi}_M = b(u) \tag{3.8L}$$

where a and b are sufficiently differentiable arbitrary functions. But using (3.8R) with (3.2R) yields

$$\psi_0 = \frac{1}{j_1}\partial_v \frac{j_1}{j_2}\partial_v \frac{j_2}{j_3}\dots\partial_v \frac{j_{N-1}}{j_N}\partial_v(j_N a) \tag{3.9R}$$

while using (3.8L) with (3.2L) gives

$$\bar{\psi}_0 = \frac{1}{\bar{j}_{-1}}\partial_u \frac{\bar{j}_{-1}}{\bar{j}_{-2}}\partial_u \frac{\bar{j}_{-2}}{\bar{j}_{-3}}\dots\partial_u \frac{\bar{j}_{M+1}}{\bar{j}_M}\partial_u(\bar{j}_M b). \tag{3.9L}$$

But then $w_0 \sim (\partial_v j_0 \partial_u - j_1)\psi_0 = 0$ is solved by ψ_0, and by $\bar{j}_0\bar{\psi}_0$ according to (3.5L), so that the general solution of w_0 is given by $\psi_0+\bar{j}_0\bar{\psi}_0$. Equivalently, the general solution of $\bar{w}_0$ is given by $j_0\psi_0+\bar{\psi}_0$. But the support of ψ_0 with respect to the coordinate v is clearly no greater than that of $a(v)$, while the support of $\bar{\psi}_0$ with respect to u is clearly no greater than that of $b(u)$. It easily follows that w_0 and $(\bar{w}_0)$ have the CPP. We thus have the general result that a doubly terminating substitution sequence w corresponds to a w_0 with the CPP. Actually every element w_n in W obviously has the property and it is a property of the sequence w and of course of $R_w(w)$). Similar statements apply if we begin with a doubly terminating $\bar{w}$ in $\bar{W}$.

Suppose we have a w in W that merely terminates to one side, say to the right. Half of the preceding argument will apply yielding 'one half' of a general solution consistent with the CPP, but the valid implication that $\bar{w}$ *also* terminates to the *right* yields a solution to $\bar{w}_0$, and thus to w_0, in which the derivatives of (3.9L) are replaced by integrals. This is *not* consistent with the CPP. Of course, if w terminates to neither side the CPP is entirely lost. Thus it is likely that double termination is essential for the CPP. Of course, this discussion applies directly only to equations in a normal form, i.e. to elements of *WE* or $\overline{WE}$. It is, however, made clear in KN that elements in LSWE have the CPP if and only if their representations in *WE* and $\overline{WE}$ have it. Thus the criterion just obtained extends to the set LSWE.

Before proceeding to search for all doubly terminating substitution sequences a few remarks are in order concerning the contents of this section and KN. The formal development given here differs from that in KN in several respects. First of all we have found it useful to introduce somewhat more notation than they did. This is primarily because we have distinguished between left and right substitution sequences, which they did not. Our reason for doing this was to prevent those elements of sequences indexed by a 0 from appearing to have a significance that they do not have. The latter point was of particular concern to us because the bijections $B: M \to W$, $\bar{B}: M \to \bar{W}$ to be defined in § 4 will involve entire substitution sequences, with no special status attributed to any one element.

4. The bijections between M, and W and $\bar{W}$

The set of motions of the infinite Toda lattice of strings, M, comprises the set of sequences of functions $\{y_n|n\in Z\}$ subject to (2.6). Similarly, the finite sequences of y_n subject to (2.7) and (2.9) are the particular motions for the finite Toda lattices of strings with end strings free. It is convenient to let

$$M\equiv\{\{y_n|n\in Z\}\} \tag{4.1}$$

include both types; the distinction between the infinite and finite lattices will be clear from context.

The wave equations of WE and $\overline{WE}$ have been divided into sets of equivalence classes W and $\bar{W}$ and each equivalence class w in W, or $\bar{w}$ in $\bar{W}$, can be identified by means of $w_n\sim j_n$, $\bar{w}_n\sim\bar{j}_n$, with the sequence of functions $\{j_n|n\in Z\}$, or $\{\bar{j}_n|n\in Z\}$, subject to (3.2R) and (3.2L), respectively. Let us define related sequences $\{\sigma_n|n\in Z\}$, $\{\bar{\sigma}_n|n\in Z\}$ by

$$j_{n+1}/j_n=\exp(-\sigma_n) \tag{4.2R}$$

$$\bar{j}_n/\bar{j}_{n+1}=\exp(-\bar{\sigma}_n). \tag{4.2L}$$

It follows from (3.2R) that

$$\partial^2_{vu}\sigma_n=-[-\exp(-\sigma_{n-1})+2\exp(-\sigma_n)-\exp(-\sigma_{n+1})]. \tag{4.3R}$$

Similarly it follows from (3.2L) that

$$\partial^2_{uv}\bar{\sigma}_n=-[-\exp(-\bar{\sigma}_{n-1})+2\exp(-\bar{\sigma}_n)-\exp(-\bar{\sigma}_{n+1})]. \tag{4.3L}$$

It is immediately clear, from inspection, that the σ_n satisfy the same equations as do the r_n of (2.6) and that the j_n therefore satisfy the same equations as do the y_n of (2.5). Consequently the bijection

$$B:M\to W:\{y_n|n\in Z\}\to\{j_n|n\in Z\}\to\{w_n|n\in Z\} \tag{4.4R}$$

carries any motion on the Toda lattice of strings to a substitution sequence of j and (equivalent) w, and vice versa, where from (4.2R) we may take

$$j_n=\exp(-y_n)\qquad n\in Z. \tag{4.5R}$$

Similarly, if we define

$$\bar{B}:M\to\bar{W}:\{y_n|n\in Z\}\to\{\bar{j}_n|n\in Z\}\to\{\bar{w}_n|n\in Z\} \tag{4.4L}$$

it carries any motion on the Toda lattice of strings to a substitution sequence of $\bar{j}$ and (equivalent) $\bar{w}$, and vice versa, where from (4.2L) we may take

$$\bar{j}_n=\exp(y_n)\qquad n\in Z. \tag{4.5L}$$

The preceding rests on the fact that the $\bar{\sigma}_n$ also satisfy (2.6), while (3.2L) differs from (3.2R) in such a way that there is a sign difference between (4.5R) and (4.5L). The properties and some applications of (4.4) and (4.5) will concern us in the remainder of the paper.

One general property is immediate. Suppose $y\equiv\{y_n|n\in Z\}$ in M is carried by B to $\{w_n|n\in Z\}$ in W and thus via $w_n\sim j_n$ to $\{\exp(-y_n)|n\in Z\}$ in J. Then the same y is carried by $\bar{B}$ to the reciprocal series $\{\exp(y_n)|n\in Z\}$ in $\bar{J}$; in terms of wave equations the corresponding nth elements $w_n=B(y_n)$ in WE and $\bar{w}_n=\bar{B}(y_n)$ in $\overline{WE}$ are right and left normal forms of one another. Further properties will be germane to the calculations in subsequent sections.

5. All doubly terminating substitution sequences

It was established in § 3, following KN, that an element in LSWE has the CPP if its representative in *WE* (or equivalently in $\overline{WE}$) generates a doubly terminating substitution sequence, and it is likely that the converse holds. But such a sequence is defined by $1/j_{M-1}=0$ for some $M<0$ and $j_{N+1}=0$ for some $N>0$, in the notation of § 3. According to (4.5R) this means $y_{N+1}=-y_{M-1}=+\infty$, which is precisely a motion on a Toda lattice of $M+N+1$ strings with the end strings free, according to § 2. If we index so that $M=1$, the motion is given by $y_1,\ldots,y_N$ satisfying (2.7) and (2.9). But the set of all such $y_1,\ldots,y_N$ is exactly what is given by (2.20). In terms of the j, the most general sequence of the form $1/0, j_1,\ldots,j_{N-1}, j_N, 0$ is thus given by

$$
\begin{aligned}
j_1 &= -[\varphi_0(u)\Psi_0(v)]^{-1}\Delta_1 X_{N-1}/\Delta_0 X_{N-1}\\
&\ \vdots\\
j_n &= (-1)^n[\varphi_0(u)\Psi_0(v)]^{-n}\Delta_n X_{N-1}/\Delta_{n-1}X_{N-1}\\
&\ \vdots\\
j_N &= (-1)^N[\varphi_0(u)\Psi_0(v)]^{-N}\Delta_N X_{N-1}/\Delta_{N-1}X_{N-1}
\end{aligned}
\tag{5.1}
$$

where X_{N-1} is defined in terms of $2N-2$ arbitrary one-variable functions by (2.16). It seems worth emphasising that the arbitrary functions $\varphi_0(u)$, $\Psi_0(v)$, which were injected into (2.20) by a transformation, can similarly be eliminated from (5.1) by a transformation. The remaining degrees of freedom in (2.20), $\varphi_1,\ldots,\varphi_{N-1}$, $\Psi_1,\ldots,\Psi_{N-1}$, parametrise in a non-trivial way a remarkably wide class of wave equations with the CPP.

6. Specialisation to self-adjoint wave equations

It is now necessary to return, briefly, to the general formalism laid out in § 3. Given an element $j=\{j_n|n\in Z\}$ in J let us define its *adjoint* by

$$\mathrm{Ad}: J\to\bar{J}:\{j_n|n\in Z\}\to\{\bar{j}_n|n\in Z\} \tag{6.1R}$$

with $j_n=\bar{j}_{-n}$, and similarly

$$\overline{\mathrm{Ad}}:\bar{J}\to J:\{\bar{j}_n|n\in Z\}\to\{j_n|n\in Z\} \tag{6.1L}$$

where $\bar{j}_n=j_{-n}$. The above definitions appear to give some special status to the element in a sequence indexed by 0, but no special status is intended. Now, however, we define a sequence j or $\bar{j}$ to be *self-adjoint* when

$$\mathrm{Ad}(j)=R_j(j) \qquad j_{-n}=1/j_n \tag{6.2R}$$

$$\overline{\mathrm{Ad}}(\bar{j})=\bar{R}_j(\bar{j}) \qquad \bar{j}_{-n}=1/\bar{j}_n \tag{6.2L}$$

respectively. This definition means that

$$j_0=1 \qquad j_n=1/j_{-n} \tag{6.3R}$$

$$\bar{j}_0=1 \qquad \bar{j}_n=1/\bar{j}_{-n} \tag{6.3L}$$

which certainly does single out the elements j_0 in j and $\bar{j}_0$ in $\bar{J}$. But it is easily checked that, as stated in KN, (6.3R) is equivalent to $w_0\sim(\partial_v j_0\partial_u-j_1)\Psi_0=0$ being a self-adjoint wave equation, and likewise for (6.3L). It is clear that other w_n in w are not self-adjoint in this case, although they are special in that they are 'Darboux equivalent' to a self-adjoint equation w_0; similar remarks hold with reference to $\bar{w}_0$ and $\bar{w}$.

In the particular case where we ask for substitution sequences that are doubly terminating and self-adjoint we can index so that the sequence takes the form

$$1/0, j_{-P}, \ldots, j_{-1}, 1, j_1, \ldots, j_P, 0 \tag{6.4}$$

where, of course, $j_{-n} = 1/j_n$. It is far from obvious how (5.1) is to be specialised to achieve (6.4). However this case was worked out by Torrence (1986), by using non-trivial results on (2.14)-(2.16) obtained by Leznov (1980). It turns out that

$$\begin{aligned} j_1 &= (-1)^{P-1}\Delta_{P-1}X_P/\Delta_P X_P \\ &\vdots \\ j_n &= (-1)^{P-n}\Delta_{P-n}X_P/\Delta_{P-n+1}X_P \\ &\vdots \\ j_P &= \Delta_0 X_P/\Delta_1 X_P \end{aligned} \tag{6.5}$$

where

$$X_P = \sum_{k=0}^{2P} (-1)^k I^k(u) I^k(v)[\varphi_1(u)\ldots\varphi_P(u)\Psi_1(v)\ldots\Psi_P(v)]^{-1} \tag{6.6}$$

and

$$\begin{aligned} I^k(u) &\equiv \int^u \varphi_1\,\mathrm{d}u_1 \int^{u_1} \varphi_2\,\mathrm{d}u_2 \ldots \int^{u_{P-1}} \varphi_P\,\mathrm{d}u_P \int^{u_P} \varphi_P\,\mathrm{d}u_{P+1} \int^{u_{P+1}} \varphi_{P-1}\,\mathrm{d}u_{P+2}\ldots \\ &\quad \times \int^{u_{2P-2}} \varphi_2\,\mathrm{d}u_{2P-1} \int^{u_{2P-1}} \varphi_1\,\mathrm{d}u_{2P} \\ I^k(v) &\equiv \int^v \Psi_1\,\mathrm{d}v_1 \int^{v_1} \Psi_2\,\mathrm{d}v_2 \ldots \int^{v_{P-1}} \Psi_P\,\mathrm{d}v_P \int^{v_P} \Psi_P\,\mathrm{d}v_{P+1} \int^{v_{P+1}} \Psi_{P-1}\,\mathrm{d}v_{P+2}\ldots \\ &\quad \times \int^{v_{2P-2}} \Psi_2\,\mathrm{d}v_{2P-1} \int^{v_{2P-1}} \Psi_1\,\mathrm{d}v_{2P} \end{aligned} \tag{6.7}$$

with $j_{-n} = 1/j_n$ generating $j_{-1}, \ldots, j_{-P}$ from (6.5).

The image of the self-adjointness of a substitution sequence $J = \{w_n | n \in Z\}$ under B^{-1} is obvious and physically simple. According to (4.5) $j_0 = 1$ corresponds to $y_0 = 0$, i.e. to a fixed string, and $j_n = 1/j_{-n}$ corresponds to $y_n = -y_{-n}$. Thus a self-adjoint substitution sequence corresponds to a motion of the Toda lattice of strings that is antisymmetrical about a fixed string. If we add the requirement of double termination, i.e. free end strings, then clearly we must have an odd number of strings with the middle string fixed.

Note that 'gauge freedom' of the type given by (2.19) is inapplicable as it would lead to a violation of $j_0 = 1$, and it is for this reason that the functions $\Psi_0(u)$, $\Psi_0(v)$ that appear in (2.20) play no role in (6.5)-(6.7).

7. Conclusion

It was suggested by Torrence (1986) that those elements of LSWE with the CPP are the natural candidates to be called non-dispersive wave equations. That suggestion was made in the context of the set of self-adjoint wave equations, but it seems equally appropriate in the more general setting of § 5. In this sense, the maps defined in § 4 have been applied in § 5 to find probably all non-dispersive linear wave equations in $1+1$ dimensions. Regardless of the acceptability of the suggested definition, the CPP is well defined, and clearly a significant property for a wave equation to have, and to delineate the class as constructively as was done in § 5 seems well worthwhile.

It is actually quite possible that the results of § 4 have much more to offer. Among the various Toda lattices and their motions are a variety of well structured special cases. For example, the periodic Toda lattices do not correspond under B to wave equations with the CPP, but they do correspond to a very restricted family of wave equations and it would be of interest to know what distinguishes them, as wave equations, from other elements of LSWE. In addition, there are special motions on some Toda lattices, for example motions described as solitons. Do these motions map under B to wave equations of special interest? Reversing the process, one might consider elements of LSWE that are special as wave equations, put them into a normal form, generate a substitution sequence and map it by B^{-1} to a particular motion on a particular Toda lattice. A particularly interesting question along these lines concerns the images under a generalised B^{-1} of higher-order wave equations of various types. These many questions are under active investigation.

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